Phys 410
Fall 2014
Lecture #15 Summary
21 October, 2014

We started by looking at the pendulum whose point of suspension is forced to rotate on a circle of radius R at a fixed angular velocity ω. The key step is to write down the (x, y) coordinates of the bob in terms of a minimum number of parameters and generalized coordinates. We did this by describing the location of the bob starting from the center axis of the circle (chosen to be the origin) and describing the location of the point of suspension, and then adding the vector position of the bob relative to the point of suspension. The location of the particle is specified by a single variable, φ, which describes the deviation of the bob from the vertical. This location (x, y) was then differentiated with respect to time to get the vector velocity (in terms of φ, $\dot{\varphi}$, and time t), and the kinetic energy was constructed from that. The potential energy is entirely due to gravity, so the Lagrangian can be constructed. The Euler-Lagrange equation yields the equation of motion for the single generalized coordinate φ. The resulting motion can be quite complicated.

If a generalized coordinate does not appear in the Lagrangian it is said to be *ignorable* or *cyclic*. The corresponding generalized momentum is conserved. This leads to simplifications in the description of the motion.

We derived a new quantity known as the Hamiltonian. The Lagrangian was engineered specifically to reproduce Newton's second law in component form, however it does not have a simple physical interpretation. By taking the total time derivative of the Lagrangian ($\frac{d\mathcal{L}}{dt}$) we could create a new quantity $\mathcal{H}$ that is time-invariant, subject to the condition that $\frac{\partial \mathcal{L}}{\partial t} = 0$ (i.e. that the Lagrangian has no explicit time dependence), and it is found to be $\mathcal{H} = \sum_{i=1}^{n} p_i \, \dot{q}_i - \mathcal{L}$, where $p_i = \partial\mathcal{L}/\partial\dot{q}_i$. If, in addition, there is a time-independent relationship between the Cartesian coordinates and the generalized coordinates, $\vec{r}_\alpha = \vec{r}_\alpha(q_1, q_2, \dots q_i, \dots q_n)$, then the Hamiltonian has a simple interpretation as the total mechanical energy $T + U$.

The process of "doing quantum mechanics" proceeds as follows. Start with the classical Lagrangian for the problem. Derive the Hamiltonian for the case of a time-independent Lagrangian, and express the Hamiltonian in terms of just the coordinates and their conjugate momenta (calculated as $p_i = \partial\mathcal{L}/\partial\dot{q}_i$). Proceed to quantize this Hamiltonian (see the Wiki page on Canonical Quantization, or Chapter IV of Dirac's *The Principles of Quantum Mechanics*).

We considered the two-body problem of two objects interacting by means of a conservative central force, with no other external forces acting. This problem eventually simplifies from that of 6 degrees of freedom (for 2 particles in three dimensions) to essentially a single particle moving in one dimension! The Lagrangian can be simplified by adopting the generalized coordinates: relative coordinate $\vec{r} = \vec{r}_1 - \vec{r}_2$, and the center of mass coordinate $\vec{R} = (m_1\vec{r}_1 + m_2\vec{r}_2)/M$, where $M = m_1 + m_2$ is the total mass. The two-particle Lagrangian simplifies to $\mathcal{L}(\vec{R},\vec{r}) = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$, where $\mu = m_1m_2/M$ is called the reduced mass because it is smaller than either m_1 or m_2. Because this Lagrangian is independent of $\vec{R}$, it means that the center of mass (CM) momentum $M\dot{\vec{R}}$ is constant. The other Lagrange equation gives $\mu\ddot{\vec{r}} = -\vec{\nabla}U(r)$, which is Newton's second law for the relative coordinate.

Taking advantage of the CM conserved momentum, we can jump to the CM (inertial) reference frame, where the CM is at rest, and the two particles are always moving with equal and opposite momenta (this follows from the fact that $\dot{\vec{R}} = 0$ in the CM reference frame). In this reference frame, the Lagrangian simplifies to $\mathcal{L} = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$. Because only central forces act, the net torque that the particles exert on each other is zero, hence the total angular momentum of the particles ($\vec{L}$) as seen in this reference frame is conserved. Writing the sum of the angular momenta of the two particles, as seen in the CM reference frame, in terms of the generalized coordinates, we find $\vec{L} = \vec{r} \times \mu\dot{\vec{r}}$, which is the same as the angular momentum of a single particle of mass μ. Because $\vec{L}$ is conserved (including its direction), the vectors $\vec{r}$ and $\dot{\vec{r}}$ must remain in a fixed two-dimensional plane throughout the motion. This means that the motion is strictly two-dimensional! Note that purely 2D motion (and the concept of a trajectory) is prohibited in quantum mechanics, hence the reduced mass particle in the hydrogen atom problem spreads into a "cloud" of probability density, very roughly speaking.

Now we have to solve the remaining two-dimensional motion problem with this Lagrangian: $\mathcal{L} = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$. Going over to polar coordinates for $\vec{r}$, we get $\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r)$, as derived earlier. There are two Lagrange equations that follow from this Lagrangian. First we note that φ is an ignorable coordinate, hence the angular momentum of the 'particle' is conserved: $\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \mu r^2\dot{\varphi} = constant$. This is in fact just the z-component of the total angular momentum vector $\vec{L}$ that we calculated above. We give it a new name, ℓ, because it is a constant of the motion (you may now recognize the notation from the quantum treatment of the Hydrogen atom). The other Lagrange equation (for r) gives $\mu\ddot{r} = \mu r\dot{\varphi}^2 - dU/dr$. The first term on the RHS is the centrifugal force for the 'particle'.